MTH 304 Final Solutions

1. Consider the multiplicative matrix groups

$$\mathrm{GL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a,b,c,d) \in \mathbb{R}^4 \text{ and } ad - bc \neq 0 \right\}$$

and

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a,b,c,d) \in \mathbb{R}^4 \text{ and } ad - bc = 1 \right\}.$$

(a) Show that $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ are topological groups.

(b) Show that $\mathrm{GL}(2,\mathbb{R})$ is homeomorphic to an open subspace of $\mathbb{R}^4$, while $\mathrm{SL}(2,\mathbb{R})$ is homeomorphic to a closed subspace of $\mathbb{R}^4$. [Hint: Consider the determinant map $\mathrm{Det} : \mathbb{R}^4 \to \mathbb{R}$.]

(c) Show that $\mathrm{GL}(2,\mathbb{R})$ is not connected, and $\mathrm{SL}(2,\mathbb{R})$ is noncompact.

Solution. (a) As both G and H are subsets of $\mathbb{R}^4$, they inherit the subspace topology from the standard topology in the ambient space. We know from MTH 301 that both $G = \mathrm{GL}(2,\mathbb{R})$ and $H = \mathrm{SL}(2,\mathbb{R})$ are multiplicative groups, and $H \trianglelefteq G$. So it suffices to show that G is a topological group. In other words, we need to establish that the matrix product operation

$$\varphi : G \times G \to G \, : \, (A,B) \stackrel{\varphi}{\mapsto} AB,$$

and the matrix inversion operation

$$I : G \to G \, : \, A \stackrel{I}{\mapsto} A^{-1}$$

are continuous maps. We know that for two matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \, B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in G,$$

we have

$$\varphi(A,B) = AB = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix},$$

and for $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have

$$I(C) = C^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Viewing φ as a map

$$\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4 \,:\, (A, B) \overset{\varphi}{\mapsto} ((AB)_{11}, (AB)_{12}, (AB)_{21}, (AB)_{22}),$$

(where $A = (a_1, b_1, c_1, d_1), B = (a_2, b_2, c_2, d_2) \in \mathbb{R}^4$) we see that its four component functions of the form $(A, B) \mapsto (AB)_{ij}$ are multivariable polynomials, and hence φ is continuous. Similarly, viewing I as a function $\mathbb{R}^4 \to \mathbb{R}^4$, we see that its four components functions are rational functions (i.e. polynomial/polynomial), which have the a common denominator $ad - bc \neq 0$. Hence, I is a continuous function, and this shows that G is a topological group.

(b) Now consider the determinant map

$$\text{Det} : G \to \mathbb{R} \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\text{Det}}{\longmapsto} ad - bc.$$

Once again, viewing Det as a map $\mathbb{R}^4 \to \mathbb{R}$, we see that as $ad - bc$ is a polynomial, and so Det is continuous. It is apparent that as $\mathbb{R}$ is a T_1 space,

$$A = \text{Det}^{-1}(\mathbb{R} \setminus \{0\}) \text{ and } \text{Det}^{-1}(\{1\}),$$

from which (b) follows.

(c) The disconnectedness of G follows from the fact that the open subsets

$$\text{Det}^{-1}((-\infty, 0)) \text{ and } \text{Det}^{-1}((0, \infty))$$

form a separation for G. To see the noncompactness of H, it suffices to show that H is unbounded under the standard metric in $\mathbb{R}^4$ (by the Heine-Borel property). For any $n \in \mathbb{N}$, consider the matrix $A_n \in H$ defined by

$$A_n = \begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix}.$$

Then $\|A_n\| = \sqrt{n^2 + 1/n^2}$, and

$$\lim_{n \to \infty} \|A_n\| = \infty,$$

which shows that $\{A_n \,|\, n \in \mathbb{N}\}$ is an unbounded subset of H. As H has an unbounded subset, it is unbounded.

2. Show that if X is separable, then every collection of disjoint open sets in X is countable.

   Solution. Let A be a countable dense subset of X such that $\bar{A} = X$. Let $\{U_\alpha\}_{\alpha \in J}$ be an arbitrary collection of disjoint open sets in X. Since $\bar{A} = X$, for each α, there exists an $x_\alpha \in A \cap U_\alpha$. Moreover, the fact that the open sets in $\{U_\alpha\}_{\alpha \in J}$ are mutually disjoint implies that $x_\alpha \neq x_\beta$, whenever $\alpha \neq \beta$. As A is countable, $Y = \{x_\alpha \,|\, \alpha \in J\}$ is a countable subset of X. Since Y is bijective with J, J has to be countable.

3. (a) Define the one-point compactification of a locally compact Hausdorff space.

   (b) Show that the open point compactification of $\mathbb{N}$ is homeomorphic to $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$.

   Solution. (a) See 1.2 (xxx) in the Lesson Plan.

   (b) Let $K = \{1/n \,|\, n \in \mathbb{N}\}$. The inversion map

$$\iota : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\} \; : \; x \overset{\iota}{\mapsto} 1/x$$

is a homeomorphism, as it is a rational function. This shows that

$$\iota|_{\mathbb{N}} : \mathbb{N} \to K$$

is a homeomorphism. As K is a closed subset of the locally compact Hausdorff space $\mathbb{R}$, K is locally compact. Hence, it follows from 1.2 (xxxi) that K has a one-point compacitification. But that fact that $\overline{K} = K \cup \{0\}$ is compact space (being a closed and bounded subspace of $\mathbb{R}$) and Hausdorff implies that $\overline{K}$ is the unique one point compactification of K, up to homeomorphism.

Let $X^* = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $X = \mathbb{N}$. Then by defining $\iota(\infty) = 0$, the map ι extends to a bijective map

$$\hat{\iota} : X^* \to Y,$$

where $Y = K \cup \{0\}$. Since X^* compact and Y is Hausdorff, it suffices to show that $\hat{\iota}$ is continuous, and in particular, $\hat{\iota}$ is continuous at ∞. Let U be a neighborhood of 0 in Y. Then by definition $Y \setminus U$ is compact, which implies that $\hat{\iota}^{-1}(Y \setminus U) = \iota^{-1}(K \setminus U)$ is compact, and so

$$\hat{\iota}^{-1}(U) = X \setminus \hat{\iota}^{-1}(Y \setminus U)$$

is a open neighborhood of ∞, which is mapped into U. This shows that $\hat{\iota}$ is continuous, and hence a homeomorphism.

4. Let X be a nonempty compact Hausdorff space without isolated points.

   (a) Show that for each nonempty open $U \subset X$, and $x \in X$, there exists an open set $V \subset U$ such that $x \notin \overline{V}$.

   (b) Show that there exists no surjective map $f : \mathbb{N} \to X$. [Hint: Consider $x_1 = f(1)$ and $U = X$ and apply (a) to get a V. Now take $x_2 = f(2)$ and $U = V$, and so on. Finally, use the finite intersection property.]

   Solution 1. The argument is analogous to the one used in the proof of Theorem 27.7 (Page 174) in Munkres.

5. If every real-valued continuous function on a metric space X is bounded, then show that X is compact. [Hint: If X is not compact, consider a sequence (x_n) with no covergent subsequence. Consider the map $x_n \mapsto n$, and use the Tietze extension theorem.]

   Solution. We know that a metric space is compact iff its sequentially compact. Suppose that X is not compact. Then X is not sequentially compact, which implies that there exists a sequence (x_n) in X that has no convergent subsequence. This implies that the set $A = \{x_n \mid n \in \mathbb{N}\}$ has no limit points, and so A is closed in X. Define a function

   $$f : A \to \mathbb{N}(\subset \mathbb{R}) \, : \, x_n \overset{f}{\mapsto} n.$$

   Then clearly, f is a continuous, as its a bijective map between two discrete subspaces. Moreover, as X is metrizable, it is normal, and by the Tietze's extension theorem, f extends to a continuous map

   $$\hat{f} : X \to \mathbb{R},$$

   which is clearly unbounded.

6. Consider the standard quotient map $q : \mathbb{R}^2 \to S^1 \times S^1 (\approx \mathbb{R}^2/\mathbb{Z}^2)$ induced by the equivalence relation $\sim$ on $\mathbb{R}^2$ defined by

   $$(x_1, y_1) \sim (x_2, y_2) \iff (x_2 - x_1, y_2 - y_1) \in \mathbb{Z}^2.$$

   Consider the multiplicative matrix group

   $$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a, b, c, d) \in \mathbb{Z}^4 \text{ and } ad - bc = 1 \right\}.$$

For a fixed matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, define a map $M_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$M_A((x, y)) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

for any $(x, y) \in \mathbb{R}^2$.

(a) Show that M_A induces a map $\widetilde{M}_A : S^1 \times S^1 \to S^1 \times S^1$, which is a homeomorphism.

(b) Show that for a fixed $A \in \mathrm{SL}(2, \mathbb{Z})$, we have $q \circ M_A = \widetilde{M}_A \circ q$.
Solution. (a) & (b) We know that by definition, $q((x, y)) = [(x, y)]$, where $[(x, y)]$ denotes the equivalence class of (x, y) under $\sim$, given by

$$[(x, y)] = \{(x + p, y + q) \mid p, q \in \mathbb{Z}\}.$$

We define

$$\widetilde{M}_A : S^1 \times S^1 \to S^1 \times S^1$$

by

$$\widetilde{M}_A([(x, y)]) = [M_A((x, y))], \text{ for all } [(x, y)] \in S^1 \times S^1.$$

Then $\widetilde{M}_A$ is well-defined, for if $[(x, y)] = [(x', y')]$, then there exists $p, q \in \mathbb{Z}$ such that $(x + p, y + q) = (x', y')$, so that

$$\begin{aligned}
\widetilde{M}_A([(x', y')]) &= [M_A(x', y')] \\
&= [M_A(x + p, y + q)] \\
&= [a(x + p) + b(y + q), c(x + p) + d(y + q)] \\
&= [M_A(x, y) + M_A(p, q)] \\
&= [M_A(x, y)] \\
&= \widetilde{M}_A([(x, y)].
\end{aligned}$$

Moreover, for all $(x, y) \in \mathbb{R}^2$, we have that

$$\begin{aligned}
(q \circ M_A)((x, y)) &= q(M_A((x, y)) \\
&= [M_A((x, y))] \\
&= \widetilde{M}_A([x, y]) \\
&= \widetilde{M}_A(q((x, y)) \\
&= (\widetilde{M}_A \circ q)((x, y)).
\end{aligned}$$

This shows that the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{M_A} & \mathbb{R}^2 \\ \big\downarrow p & & \big\downarrow p \\ S^1 \times S^1 & \xrightarrow{\widetilde{M}_A} & S^1 \times S^1 \end{array} \qquad (*)$$

is commutative.

Since $A \in \mathrm{SL}(2, \mathbb{Z})$, M_A is an invertible linear map, hence a homeomorphism. Now for any $[(x', y')] \in S^1 \times S^1$, there exits $(x, y) \in \mathbb{R}^2$ such that $M_A((x, y)) = (x', y')$, which implies that

$$\widetilde{M}_A([(x, y)]) = [M_A((x, y))] = [(x', y')],$$

and so it follows that $\widetilde{M}_A$ is surjective. Moreover, we have that

$$\begin{aligned} \widetilde{M}_A([(x, y)]) = \widetilde{M}_A([(x', y')]) &\implies [M_A((x, y))] = [M_A((x', y'))] \\ &\implies (\widetilde{M}_A \circ q)((x, y)) = (\widetilde{M}_A \circ q)((x', y')) \\ &\implies (q \circ M_A)((x, y)) = (q \circ M_A)((x', y')) \\ &\implies [M_A((x, y))] = [M_A((x', y'))] \\ &\implies [(x, y)] = [(x', y')], \end{aligned}$$

from which the injectivity of $\widetilde{M}_A$ follows.

Since q is a open and continuous map (why?), for an open set $U \subset S^1 \times S^1$, we have that $q^{-1}(U)$ is open in $\mathbb{R}^2$, so that $M_A(q^{-1}(U)$ is open in $\mathbb{R}^2$, and so

$$(q \circ M_A \circ q^{-1})(U) = \widetilde{M}_A(U)$$

is open in $S^1 \times S^1$. Finally, for an open set V in $S^1 \times S^1$, we have that

$$\widetilde{M}_A^{-1}(V) = (q \circ M_A \circ q^{-1})^{-1}(V) = (q \circ M_A^{-1} q^{-1})(V),$$

which is open in $S^1 \times S^1$. This shows that $\widetilde{M}_A$ is a homeomorphism.

Solution 2. Alternatively, one could consider the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "q"] \arrow[dr, "q \circ M_A"] & \\ S^1 \times S^1 \arrow[r, dashed, "\widetilde{M}_A"] & S^1 \times S^1 \end{tikzcd} \qquad (**)$$

that is equivalent to the diagram (*) above. As the map $q \circ M_A$ is constant on each fiber of q (why?), by 1.10 (x) of the Lesson Plan, there exists a map $\widetilde{M_A}$ as indicated in (**), that makes the diagram commute. Consequently, (*) is also commutative. Moreover, since q and M_A are continuous maps, we have that $q \circ M_A$ is continuous, and once again, 1.10 (x) would imply that $\widetilde{M_A}$ is continuous.

The fact that q is an open map (from class) and M_A is a homeomorphism implies that $q \circ M_A$ is a surjective, continuous and open map, and hence a quotient map. Finally, by 1.10 (xi), $q \circ M_A$ will induce a homeomorphism $S^1 \times S^1 \to S^1 \times S^1$, which in this case is precisely the map $\widetilde{M_A}$.